Performance of Precoded Integer-Forcing for Closed-Loop MIMO Multicast

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***Abstract*—The integer-forcing receiver architecture has recently been proposed as a high-performance, yet low-complexity, equalization scheme, that is applicable when all data streams are encoded with the same linear code. It was further shown in [1], that this receiver architecture, when coupled with space-time linear precoding is able to achieve the capacity of the open-loop multiple-input multiple-output channel, up to a constant gap that depends only on the number of transmit antennas. The gap, however, is quite large and thus provides performance guarantees that are useful only for high values of capacity. In this work, we consider the problem of multicast over multiple-input multiple-output channels to a modest number of users, and with space-only linear precoding. It is assumed that channel state information is available to the transmitter, allowing it to optimize the precoding matrix so as to maximize the achievable transmission rate. It is numerically demonstrated that this architecture allows to very closely approach the multicast capacity at all transmission rates regimes.**

I. Introduction

The Multiple-Input Multiple-Output (MIMO) Gaussian channel has been the focus of extensive research efforts since the pioneering works of Foschini [2], Foschini and Gans [3], and Telatar [4]. The multi-user MIMO Gaussian channel, i.e., the MIMO Gaussian broadcast channel, has also been widely studied for over a decade now. See, e.g., [5] for an overview.

When a different message is to be transmitted to each of the users (the private message case), the capacity region may be achieved via dirty-paper coding, practical implementations of which have been designed. See, e.g., [6].

In this work we consider, in contrast, the MIMO multicast problem, where the transmitter sends a common message to all users. This scenario is also referred to as common message broadcast.

In contrast to the single-user case, the number of data streams, constellation size, and other transmission parameters cannot be tailored to a specific user and can only depend on capacity. This makes the task of code design for MIMO multicast challenging. The present work demonstrates that these challenges are successfully met via precoded integer-forcing (IF) combined with successive interference cancellation (SIC).

We now describe the channel model more formally. A transmitter equipped with M transmit antennas wishes to send the same message to K users, where user i is equipped with N_i antennas. Denoting by $\mathbf{H}_i$ the $N_i \times M$ channel matrix corresponding to the ith user (and by $\mathcal{H} = \{\mathbf{H}\}_{i=1}^{K}$ the set of channels), the received signal at user i is

$$\boldsymbol{y}_i = \mathbf{H}_i \boldsymbol{x} + \boldsymbol{z}, \tag{1}$$

where the input vector $\boldsymbol{x}$ is subject to the power constraint

$$\mathbb{E}(\boldsymbol{x}^H \boldsymbol{x}) \leq M \cdot \mathsf{SNR}, \tag{2}$$

and the additive noise $\boldsymbol{z}$ is a vector of i.i.d. unit variance circularly symmetric complex Gaussian random variables.We consider throughout a closed-loop scenario where the channel state information (CSI) is available at both transmission ends.

The multicast capacity is defined as the compound channel capacity of (1). It is attained by a Gaussian vector input, where the mutual information is maximized over all covariance matrices $\mathbf{Q}$ satisfying $\mathrm{Tr}(\mathbf{Q}) \leq M \cdot \mathsf{SNR}$:

$$C(\mathsf{SNR}, \mathcal{H}) = \max_{\mathbf{Q}:\mathrm{Tr}(\mathbf{Q}) \leq M \cdot \mathsf{SNR}} \min_{\mathbf{H} \in \mathcal{H}} \log\det(\mathbf{I} + \mathbf{H}^H \mathbf{Q} \mathbf{H}). \tag{3}$$

In the sequel, we study the achievable rate, denoted by $R(\mathsf{SNR}, \mathcal{H})$, for several transmission schemes. As a figure of merit we consider a scheme's *efficiency* which we define as the fraction of the compound capacity achieved

$$\eta(\mathsf{SNR}, \mathcal{H}) = \frac{R(\mathsf{SNR}, \mathcal{H})}{C(\mathsf{SNR}, \mathcal{H})}. \tag{4}$$

We present a practical transceiver architecture that is able to approach the MIMO multicast capacity for a moderate number of users using linear pre- and post- processing operations in conjunction with the recently introduced integer-forcing receiver architecture [7].

Note that multicast to a large number of users essentially reduces to transmission with no CSI (beyond target rate) when the number of users tends to infinity. In such a scenario, open-loop techniques should be used. Therefore, our main concern in this work is multicast to a moderate number of users.

There are several special cases where known linear modulation techniques achieve the multicast capacity (3) when coupled with codes designed for a scalar additive white Gaussian noise (AWGN) channel. For a single receive antenna, any number of transmit antennas, and two users, beamforming achieves the multicast capacity (see [8]). For a single receive antenna, two transmit antennas and any number of users,

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Alamouti modulation coupled with a linear precoding matrix (shaping the covariance matrix to maximize (3)) achieves the multicast capacity.

Alamouti modulation results in an equivalent scalar channel with no loss in mutual information (see, e.g., [9]) and is thus ideal in terms of combining it with standard coding. Alamouti modulation is generalized (see, e.g., [10]) to any number of transmit antennas by Orthogonal Space-Time Block Codes (OSTBC). Unfortunately, except for the case of 1×2 MIMO channels, OSTBC modulation results in a reduced symbol rate which amounts to under utilization of the degrees of freedom afforded by the channel. Consequently, in all cases other than 1×2 channels, OSTBC modulation incurs a loss in mutual information. Nonetheless, it can be shown that at low SNR and when coupled with a linear precoding matrix (shaping the covariance matrix to be optimal), OSTBC modulation approaches the multicast capacity.

In a recent work [1], a practical transmission scheme for *open-loop* MIMO transmission was proposed, that achieves the mutual information (corresponding to a scaled identity input covariance matrix) up to a constant gap, for *any* MIMO channel having the same mutual information. The scheme utilizes only standard scalar AWGN codes in conjunction with linear pre- and post-processing and the IF receiver architecture [7]. As the open-loop scenario may be viewed as the limit of many users, the results are applicable also for the closed-loop scenario and serve as the starting point of the present work. The approach of [1] and a description of the IF receiver architecture is outlined in Section II. We note that in [1] linear preprocessing over both space and time is applied. This is essential for open-loop transmission (or when the number of users is large). In this work, in contrast, we apply space-only pre-processing.

The main weakness of the results of [1] is that the guaranteed gap to capacity is quite large and does not provide meaningful performance guarantees at moderate transmission rates.

The aim of the present work is to remedy this drawback by demonstrating numerically that in a closed-loop scenario with a moderate number of users, precoded-IF allows to closely approach the multicast capacity at all transmission rates.

II. Background: Single-User Integer-forcing equalization with SIC

For ease of notation, throughout this section, we assume without loss of generality that the input covariance matrix is the identity matrix.[1]

In [7], a receiver architecture scheme coined "integer forcing" was proposed which we next briefly describe. It is assumed that information bits are fed into M encoders, each of which uses the same scalar AWGN *linear* code. The latter produce M channel inputs (for example, x_m for the m'th antenna).[2] At the receiver, a *linear equalization* matrix $\mathbf{B}_{\text{INT}}$ is applied, where $\mathbf{B}_{\text{INT}}$ is designed such that the resulting equivalent channel $\mathbf{A} = \mathbf{B}_{\text{INT}}\mathbf{H}$ is a matrix all of whose entries are integers. This ensures that the output of the channel (without noise) after applying a modulo operation is a valid codeword. In the basic version of IF, each of the equalized streams is next passed to a standard (up to the additional element of a modulo operation) AWGN decoder which tries to decode a linear combination of codewords $v_m = \boldsymbol{a_m}^T\boldsymbol{x}$.

For IF equalization to be successful, decoding over all M subchannels should be correct. Therefore, the worst subchannel constitutes a bottleneck. When using MMSE equalization

$$\mathbf{B}_{\text{INT}} = \mathbf{A}\mathbf{H}^T\left(\frac{1}{\mathsf{SNR}}\mathbf{I} + \mathbf{H}\mathbf{H}^T\right)^{-1}, \qquad (5)$$

the input for the m'th decoder is

$$y_{\tilde{\text{eff}},\text{m}} = v_m + z_{\text{eff},\text{m}}$$

where

$$z_{\text{eff},\text{m}} = (\boldsymbol{b}_m^T\mathbf{H} - \boldsymbol{a}_m^T)\boldsymbol{x} + \boldsymbol{b}_m^T\boldsymbol{z}.$$

This means that we can define the effective SNR at the m'th subchannel as

$$\mathsf{SNR}_{\text{eff},\text{m}} = \left(\boldsymbol{a}_m^T(\mathbf{I} + \mathsf{SNR}\mathbf{H}^H\mathbf{H})^{-1}\boldsymbol{a}_m\right)^{-1},$$

and the effective SNR associated with the IF scheme as

$$\mathsf{SNR}_{\text{eff}} = \min_{m=1,\dots,M} \mathsf{SNR}_{\text{eff},\text{m}}. \qquad (6)$$

By Theorem 3 in [7], transmission with IF equalization can achieve any rate satisfying

$$R_{\text{IF}} < M\log(\mathsf{SNR}_{\text{eff}}).$$

In this paper, we consider using a linear precoder at the transmit side in conjunction with IF at the receiver side. We further consider a generalized version of the IF equalizer that incorporates also SIC.[3] We will demonstrate its operation via an example.

For our purposes, it will suffice to state the achievable rates of IF-SIC and an operational description of its elements. The reader is referred to [11] for the derivation, details and proofs. Denoting by $\mathbf{L}$ the following Choleskey decomposition

$$\mathbf{K}_{\boldsymbol{z}_{\text{eff}}\boldsymbol{z}_{\text{eff}}} = \mathbf{A}\left(\mathbf{I} + \mathsf{SNR}\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T = \mathbf{L}\mathbf{L}^T \qquad (7)$$

and denoting by $\ell_{m,m}$ the diagonal entries of $\mathbf{L}$, the achievable rate with IF-SIC is [11]

$$R_{\text{IF}-\text{SIC}} < M\max_{\mathbf{A}}\min_{m=1,\dots,M}\log\left(\frac{\mathsf{SNR}}{\ell_{m,m}^2}\right). \qquad (8)$$

[1] We may do so since the covariance shaping matrix $\mathbf{Q}^{1/2}$ may be absorbed into the channel by defining the effective channel $\tilde{\mathbf{H}} = \mathbf{H}\mathbf{Q}^{1/2}$. With a slight abuse of notation we use $\mathbf{H}$ to denote the effective channel.

[2] For simplicity of notation the time index is suppressed since the block length plays no role in our description. Of course, to approach capacity, one needs to use a long block length.

[3] We note that IF-SIC may in general allow using different rates per stream. We will nevertheless assume throughout that all streams are encoded via an identical linear code and hence have the same rate.

We describe the operation of the IF-SIC receiver, adopting the nomenclature of [11]. First, calculate:

1) The optimal integer matrix $\mathbf{A}$, i.e., the matrix maximizing (8).
2) The covariance matrix of the effective noise when using an IF equalizer with the matrix $\mathbf{A}$ via (7).
3) The optimal SIC matrix $\mathbf{R}$ as:

$$\mathbf{R} = \mathrm{diag}(\ell_{11}, ..., \ell_{MM}) \cdot \mathbf{L}^{-1}. \tag{9}$$

4) The optimal linear front end processing matrix $\mathbf{B}$:

$$\widetilde{\mathbf{B}_{\mathrm{INT}}} = \mathbf{R}\mathbf{A}\mathbf{H}^T \left(\frac{1}{\mathsf{SNR}} \mathbf{I} + \mathbf{H}\mathbf{H}^T \right)^{-1}. \tag{10}$$

The operation of the receiver is depicted in Figure 1, where now $\mathbf{B}_{\mathrm{INT}}$ is to be understood as $\widetilde{\mathbf{B}_{\mathrm{INT}}}$. Note that this change of linear post-processing is essential to guarantee that the resulting noise variance is minimized. The output of decoders $1, \ldots, m-1$ are multiplied by $R_{m,1}, \ldots, R_{m,m-1}$, respectively, and are then subtracted from the input to decoder m.

Example 1: Consider the following 2×2 (real) MIMO channel,

$$\mathbf{H} = \begin{bmatrix} -4.4352 & -0.4028 \\ -1.7784 & -3.5037. \end{bmatrix}$$

and assume that $\mathsf{SNR} = 1$, so that

$$C = \log(\mathbf{I} + \mathbf{H}^H \mathbf{H}) = 4 \text{ bits per real dimension.}$$

The optimal integer matrix obtained by maximizing (8) is

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Applying standard IF as specified in (5) (with an optimal front-end matrix $\mathbf{B}$ but without SIC) results in two streams with rates $r_1 = 2.1954$ and $r_2 = 1.7338$. As the achievable rate is dictated by the weaker stream, the scheme can support a rate of

$$R_{\mathrm{IF}} = 2 \times \min(r_1, r_2) = 3.4677.$$

Applying a SIC matrix

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ -0.4210 & 1 \end{bmatrix}$$

results in improved rates $r_1 = 2.1954$ and $r_2 = 1.8046$, computed via the decomposition (7) to find $l_{1,1}$ and $l_{2,2}$. A total rate

$$R_{\mathrm{IF-SIC}} = 2 \times \min(r_1, r_2) = 3.6093$$

is thus achievable. Finally, the operation $\mathbf{A}^{-1}$ (again, modulo arithmetic is assumed) is applied.

Remark 1: In a practical system, the modulo operation will likely be one-dimensional which introduces a loss of up to $\log_2\left(\frac{2\pi e}{12}\right) \approx 0.254$ bits per real degree of freedom (see, e.g., [12]). While the loss is significant for moderate values of capacity, it turns out that in this regime, the modulo operation is in fact not necessary. The performance achieved when taking into account the loss due to a one-dimensional modulo operation is addressed in Section III-C.

III. Precoded Integer-forcing Equalization for Closed-Loop Multicast

A. Problem Formulation

We evaluate the performance of IF-SIC equalization combined with optimized precoding in a closed-loop multicast setting. A single precoding matrix $\mathbf{P} = \mathbf{Q}^{1/2}\mathbf{U}$ which is composed of a covariance shaping matrix[4] $\mathbf{Q}^{1/2}$ and a unitary matrix $\mathbf{U}$ is applied at the transmitter, where $\mathbf{Q}$ is the covariance matrix achieving the multicast capacity (3). Thus, the resulting effective multicast channel is

$$\boldsymbol{y}_i = \mathbf{H}_i \mathbf{Q}^{1/2} \mathbf{U} \boldsymbol{x} + \boldsymbol{z},$$

and the achievable rate is given by

$$R_{\mathrm{P-IF-SIC}}(\mathsf{SNR}, \mathcal{H}) = \max_{\mathbf{U},\mathbf{Q}} \min_{\mathbf{H} \in \mathcal{H}} R_{\mathrm{IF-SIC}}(\mathbf{H}). \tag{11}$$

While applying a transformation $\mathbf{U}$ does not change the covariance and thus has no effect on mutual information, it *greatly impacts* the performance of IF-SIC equalization, and is therefore one of the parameters to be optimized.

There are various figures of merit that may be used to assess the performance of transmission schemes. In [1], the worst-case performance of precoded IF-SIC was analyzed. Here, we assess the performance statistically where the set of channels $\mathcal{H}$ is viewed as drawn from an ensemble of channels. Each specific set of channels results in a different compound capacity as well as in a different achievable rate. For a given scheme, we define the outage efficiency associated with the ensemble as

$$\eta_{x\%}(\mathsf{SNR}, M) = \max_{\Psi} \Pr(\eta(\mathsf{SNR}, M) < \Psi) = x\%$$

where η is defined in (4). Thus, $\eta_{0.1\%} = 0.9$, means that the evaluated scheme achieves 90% or more from the multicast capacity with a probability of 0.999.

Due to the general lack of effective closed-loop MIMO multicast transmission techniques (see Section I), for comparison purposes, we also evaluate the performance of several open-loop methods. Specifically we consider Alamouti modulation (which is the OSTBC modulation for two transmit antennas and, as mentioned above, achieves the multicast capacity at very low SNR) and precoding using the golden code [13] coupled with the IF-SIC receiver.[5]

The achievable IF-SIC rates are first presented without accounting for modulo loss (or equivalently, assuming an

[4] Without loss of generality, we define $\mathbf{Q}^{1/2}$ as the (unique) positive-semidefinite square root of $\mathbf{Q}$.

[5] As was shown in [7], standard linear receivers (zero-forcing (ZF), minimum mean square error (MMSE)), as well as their SIC variants, are special cases of IF. Therefore, by definition the performance of IF out-performs the performance of these basic linear receivers and therefore they are excluded from the comparison.

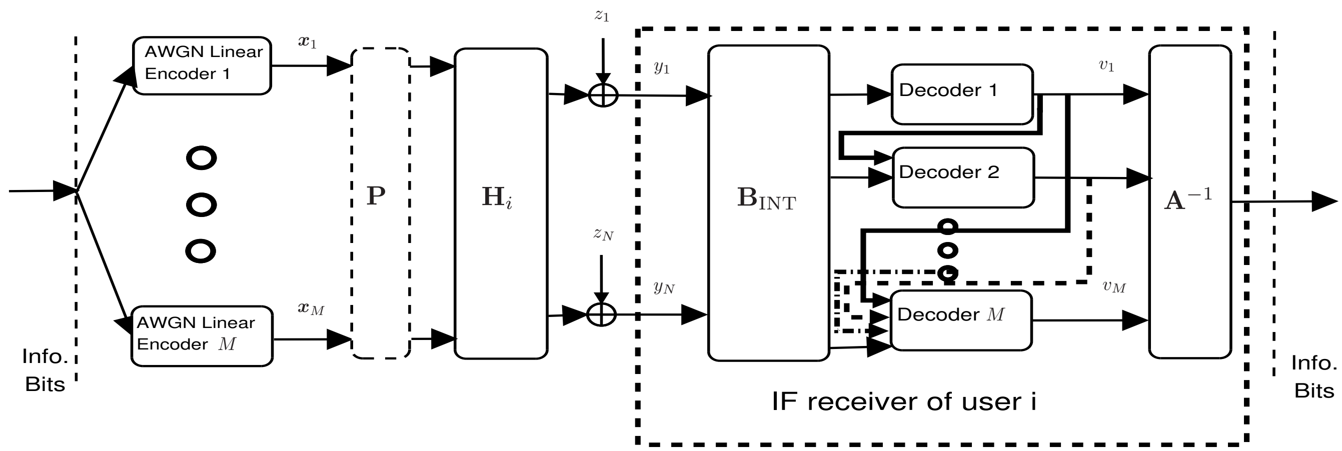

Fig. 1. Precoded-IF-SIC scheme.

optimal high-dimensional modulo operation). The impact of using a one-dimensional modulo operation is discussed in Section III-C.

B. Numerical Results

We consider a communication scenario where both the transmitter and receiver have two antennas (2×2 MIMO channels). We performed numerical optimization to find an optimal precoding matrix for the case of two or three users. We focused our attention on low and medium SNR values.

Two ensembles were considered.

Ensemble I - Rayleigh fading: all matrix entries are circularly-symmetric complex normal random variables and are drawn independently of each other. For a given SNR, each set of channels has its own multicast capacity. For this ensemble, we plot the outage efficiency as a function of SNR.

Ensemble II - Equal WI-MI with uniform distribution on singular values: the channel for each user is drawn from an ensemble of equal white-input mutual information (WI-MI). The elements in this ensemble can be described as

$$\mathbf{H} = \mathbf{V}_1 \begin{bmatrix} \tilde{\sigma_1} & 0 \\ 0 & \tilde{\sigma_2} \end{bmatrix} \mathbf{V}_2$$

where $\tilde{\sigma_i} = \sqrt{\mathsf{SNR}}\sigma_i$ (the SNR can be absorbed into the channel's singular values). The WI-MI of a single user is defined as

$$\begin{aligned} C &= \log(|\mathbf{I} + \mathsf{SNR}\mathbf{H}^H\mathbf{H}|) \\ &= \log(1 + \tilde{\sigma_1}^2)(1 + \tilde{\sigma_2}^2). \end{aligned} \quad (12)$$

For a given value of C, the ensemble is generated by drawing $\tilde{\sigma_1}^2$ uniformly in $[0, 2^C - 1]$, calculating $\tilde{\sigma_2}^2$ via (12), and then multiplying the diagonal matrix by two random matrices $\mathbf{V}_1, \mathbf{V}_2$.[6] For this ensemble, we plot the performance as a function of WI-MI.

[6]The matrices $\mathbf{V}_1, \mathbf{V}_2$ are drawn from the "circular unitary ensemble". See, e.g., [14].

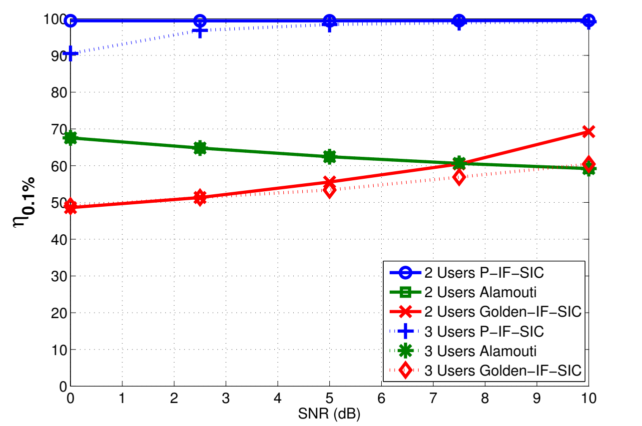

Fig. 2. Outage efficiency for 2×2 channels drawn from ensemble I with outage probability of 0.001.

Fig. 2 depicts $\eta_{0.1\%}$ for users drawn from ensemble I. We note that in the case of two users, precoded IF-SIC suffers only negligible loss w.r.t. capacity. In the three-user case, the gap is more noticeable at small values of SNR. Nonetheless, for an outage probability of 0.1%, the greatest loss in the SNR regime of interest is still no more than 10% of capacity. In both cases, precoded IF-SIC significantly outperforms the reference schemes. Fig. 3 depicts $\eta_{0.1\%}$ for users drawn from ensemble II. Finally, we tested the degradation in performance as the number of users grows. Fig. 4 shows the outage efficiency of precoded IF-SIC for ensemble II at a WI-MI of 4 bits (per complex dimension) as a function of the number of users. As expected, the loss w.r.t. the multicast capacity increases as the number of users increases. Nevertheless, even for large number of users, the 0.1% outage efficiency is much higher than the performance achievable via open-loop precoding (i.e., Alamouti and golden code precoding).

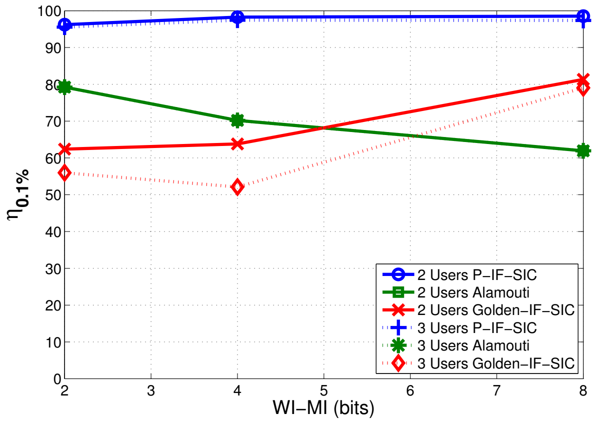

Fig. 3. Outage efficiency for 2×2 channels drawn from ensemble II with outage probability of 0.001.

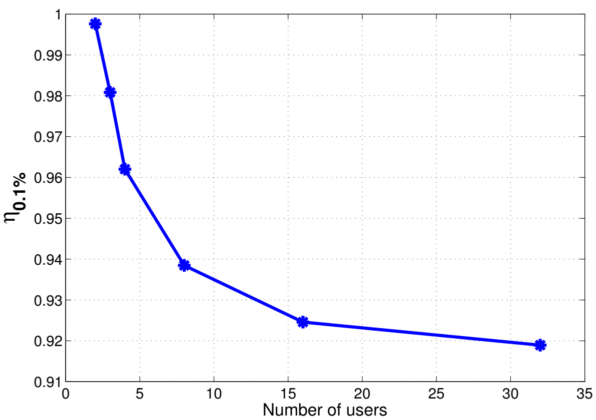

Fig. 4. Outage efficiency for different number of users, where the channels are 2×2 matrices drawn from ensemble II with outage probability of 0.001.

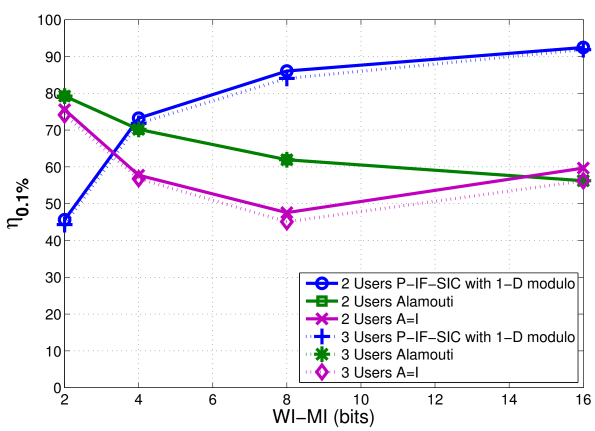

Fig. 5. Outage efficiency for 2×2 channels drawn from ensemble II with outage probability of 0.001, where a one-dimensional modulo operation is performed.

C. Effect of shaping gain

As mentioned above, the simplest implementation of IF-SIC is to use a scalar (one-dimensional) modulo operation. This results in a loss of up to 0.254 bits per real dimension. We now account for this (possible) loss, depicting the achievable rate for the WI-MI ensemble. As a reference, we consider the special case of an IF-SIC receiver which uses an equivalent channel of the form $\mathbf{A}_{\text{INT}} = \mathbf{I}$ (e.g., [15]). In this case, no modulo operation is required (since the input to the receivers is not a linear combination of codewords, but rather simply the transmitted codewords). Fig. 5 depicts the resulting performance for both methods, for the case of two users. We note that precoded IF-SIC, employing a one-dimensional modulo operation, outperforms the performance of the restricted (modulo-free) version IF-SIC for values of WI-MI greater than 4 bits per complex dimension.

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